Chapter 5

Forging Function Facts

In This Chapter

- Defining functions, domain, and range
- Identifying one-to-one functions and even vs. odd functions
- Using function composition in the difference quotient

In algebra, the word *function* is very specific. You reserve it for certain math expressions that meet the tough standards of input and output values, as well as other mathematical rules of relationships. Therefore, when you hear that a certain relationship is a function, you know that the relationship meets some particular requirements. In this chapter, you find out more about these requirements. I also cover topics ranging from the domain and range of functions to the inverses of functions, and I show you how to perform the composition of functions. After acquainting yourself with these topics, you can confront a function equation with great confidence and a plan of attack.

Describing Function Characteristics

REMEMBER

A *function* is a relationship between two variables that features exactly one output value for every input value — in other words, exactly one answer for every number inserted into the function rule.

For example, the equation $y = x^2 + 5x - 4$ is a function equation or function rule that uses the variables x and y. The x is the *input variable,* and the y is the *output variable.* If you input the number 3 for each of the x's, you get $y = 3^2 + 5(3) - 4 = 9 + 15 - 4 = 20$. The output is 20, the only possible answer. You won't get another number if you input the 3 a second time.

The single-output requirement for a function may seem like an easy requirement to meet, but you encounter plenty of strange math equations out there, so watch out.

Denoting function notation

Functions feature some special notation that makes working with them much easier. The notation doesn't change any of the properties — it just allows you to identify different functions quickly and indicate various operations and processes more efficiently.

The variables x and y are pretty standard in functions and come in handy when you're graphing functions. But mathematicians also use another format called *function notation.* For example, here are three particular functions named two different ways:

$$y = x^2 + 5x - 4 \qquad f(x) = x^2 + 5x - 4$$
$$y = \sqrt{3x-8} \qquad g(x) = \sqrt{3x-8}$$
$$y = 6xe^x - 2e^{2x} \qquad h(x) = 6xe^x - 2e^{2x}$$

On the left, you see the traditional x and y expression of the three functions. But when you see a bunch of functions written together, you can be efficient by referring to individual functions as f or g or h so listeners don't have to question what you're referring to. When I say, "Look at function g," your eyes go directly to the function I'm talking about.

Using function notation to evaluate functions

When you see a written function that uses function notation, you can easily identify the input variable, the output variable, and what you have to do to *evaluate* the function for some input (or replace the variables with numbers and simplify). You can do so because the input value is placed in the parentheses right after the function name or output value.

Evaluate $g(x) = \sqrt{3x - 8}$ when $x = 3$.

$g(3)$ is what you get when you substitute a 3 for every x in the function expression and perform the operations to get the output answer.

$g(3) = \sqrt{3(3) - 8} = \sqrt{9 - 8} = \sqrt{1} = 1$. Now you can say that $g(3) = 1$, or "g of 3 equals 1." The output of the function g is 1 if the input is 3.

Determining Domain and Range

The input and output values of a function (see the previous section) are of major interest to people working in algebra. The words *input* and *output* describe what's happening in the function (namely what number you put in and what result comes out), but the official designations for these sets of values are *domain* and *range.*

Delving into domain

The *domain* of a function consists of all the input values of the function. (Think of a king's domain of all his servants entering his kingdom.) In other words, the domain is the set of all numbers that you can input without creating an unwanted or impossible situation. Such situations can occur when operations appear in the definition of the function, such as fractions, radicals, logarithms, and so on.

Many functions have no exclusions of values, but fractions are notorious for causing trouble when zeros appear in the denominators. Radicals have restrictions as to what you can find roots of, and logarithms can only deal with positive numbers.

The way you express domain depends on what's required of the task you're working on — evaluating functions, graphing, determining a good fit as a model, to name a few. Here are some examples of functions and their respective domains:

- $\boldsymbol{f(x) = \sqrt{x - 11}}$**:** The domain consists of the number 11 and every greater number thereafter. You write this as $x \geq 11$ or, in interval notation, $[11, \infty)$. You can't use numbers smaller than 11 because you'd be taking the square root of a negative number, which isn't a real number.

- $g(x)=\frac{x}{x^2-4x-12}=\frac{x}{(x-6)(x+2)}$**:** The domain consists of all real numbers except 6 and –2. You write this domain as $x<-2$ or $-2<x<6$ or $x>6$, or, in interval notation, as $(-\infty, -2) \cup (-2, 6) \cup (6, \infty)$. It may be easier to simply write "All real numbers except $x = -2$ and $x = 6$." The reason you can't use –2 or 6 is because these numbers result in a 0 in the denominator of the fraction, and a fraction with 0 in the denominator creates a number that doesn't exist.
- $h(x) = x^3 - 3x^2 + 2x - 1$**:** The domain of this function is all real numbers. You don't have to eliminate anything, because you can't find a fraction with the potential of a zero in the denominator, and you have no radical to put a negative value into. You write this domain with a fancy R, $\Re$, or with interval notation as $(-\infty, \infty)$.

Wrangling with range

The *range* of a function is all its output values — all values you get by inputting the domain values into the *rule* (the function equation) for the function. You may be able to determine the range of a function from its equation, but sometimes you have to graph it to get a good idea of what's going on.

The following are some examples of functions and their ranges. Like domains (see the previous section), you can express ranges in words, inequalities, or interval notation:

- $k(x) = x^2 + 3$**:** The range of this function consists of the number 3 and any number greater than 3. You write the range as $k \geq 3$ or, in interval notation, $[3, \infty)$. The outputs can never be less than 3 because the numbers you input are squared. The result of squaring a real number is always positive (or if you input 0, you square 0). If you add a positive number or 0 to 3, you never get anything smaller than 3.
- $m(x)=\sqrt{x+7}$**:** The range of this function consists of all positive numbers and 0. You write the range as $m \geq 0$ or, in interval notation, $[0, \infty)$. The number under the radical can never be negative, and all the square roots come out positive or 0.

- $p(x) = \frac{2}{x-5}$**:** Some functions' equations, such as this one, don't give an immediate clue to the range values. It often helps to sketch the graphs of these functions. Figure 5-1 shows the graph of the function *p*. See if you can figure out the range values before peeking at the following explanation.

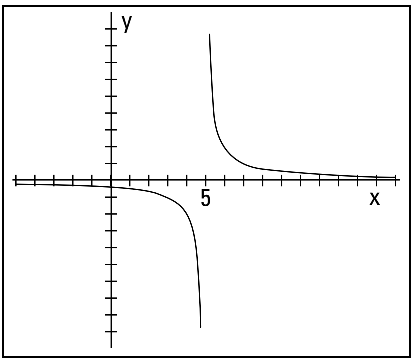

Figure 5-1: Try graphing equations that don't give an obvious range.

The graph of this function never touches the *x*-axis, but it gets very close. For the numbers in the domain bigger than 5, the graph has some really high *y* values and some *y* values that get really close to 0. But the graph never touches the *x*-axis, so the function value never really reaches 0. For numbers in the domain smaller than 5, the curve is below the *x*-axis. These function values are negative — some really small. But, again, the *y* values never reach 0. So, if you guessed that the range of the function is every real number except 0, you're right! You write the range as $p \neq 0$, or $(-\infty, 0) \cup (0, \infty)$. Did you also notice that the function doesn't have a value when $x = 5$? This happens because 5 isn't in the domain.

Counting on Even and Odd Functions

You can classify numbers as even or odd (and you can use this information to your advantage; for example, you know you can divide even numbers by 2 and come out with an integer). You can also classify some functions as even or odd.

Determining whether even or odd

An *even function* is one in which a domain value (an input) and its opposite always result in the same range value (output): $f(-x) = f(x)$ for every x in the domain. An *odd function* is one in which each domain value and its opposite produce opposite results in the range: $f(-x) = -f(x)$.

To determine if a function is even or odd (or neither), you replace every x in the function equation with $-x$ and simplify. If the function is even, the simplified version looks exactly like the original. If the function is odd, the simplified version looks like what you get after multiplying the original function equation by -1.

Show that $f(x) = x^4 - 3x^2 + 6$ is even.

Whether you input 2 or -2, you get the same output:

- $f(2) = (2)^4 - 3(2)^2 + 6 = 16 - 12 + 6 = 10$
- $f(-2) = (-2)^4 - 3(-2)^2 + 6 = 16 - 3(4) + 6 = 10$

So, you can say $f(2) = f(-2)$.

The example doesn't constitute a *proof* that the function is even; this is just a demonstration.

Show that $g(x) = x^3 - x$ is odd.

The inputs 2 and -2 give you opposite answers:

- $g(2) = (2)^3 - 2 = 8 - 2 = 6$
- $g(-2) = (-2)^3 - (-2) = -8 + 2 = -6$

So, you can say that $g(-2) = -g(2)$.

Again, I've *demonstrated,* not proved, that the function is odd.

You can't say that a function is even just because it has even exponents and coefficients, and you can't say that a function is odd just because the exponents and coefficients are odd numbers. If you do make these assumptions, you classify the functions incorrectly, which messes up your graphing. You have to apply the definitions to determine which label a function has.

Using even and odd functions in graphs

The biggest distinction of even and odd functions is how their graphs look:

- **Even functions:** The graphs of even functions are symmetric with respect to the y-axis (the vertical axis). You see what appears to be a mirror image to the left and right of the vertical axis. For an example of this type of symmetry, see Figure 5-2a, which is the graph of the even function $f(x) = \frac{5}{x^2+1}$.
- **Odd functions:** The graphs of odd functions are symmetric with respect to the origin. With this symmetry it looks the same if you rotate the graph by 180 degrees. The graph in Figure 5-2b, which is the odd function $g(x) = x^3 - 8x$, displays origin symmetry.

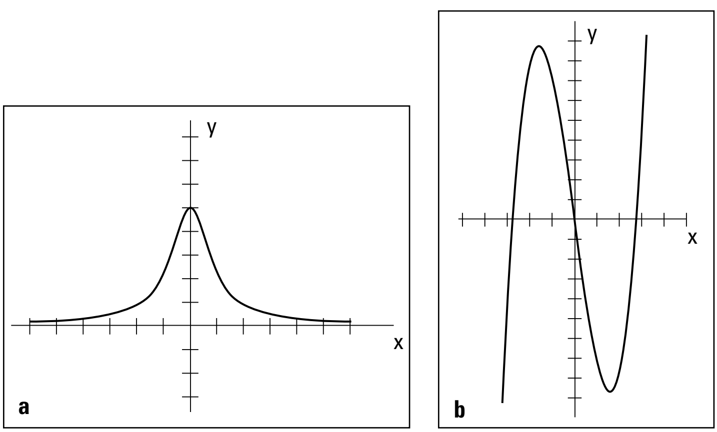

Figure 5-2: An even (a) and odd (b) function.

Taking on Functions One-to-One

Functions can have many classifications or names, depending on the situation and what you want to do with them. One very important classification is deciding whether a function is one-to-one.

Defining which functions are one-to-one

A function is *one-to-one* if you have exactly one output value for every input value *and* exactly one input value for every output value. Formally, you write this definition as follows:

> If f is a one-to-one function, then when $f(x_1) = f(x_2)$, it must be true that $x_1 = x_2$.
>
> In simple terms, if the two output values are the same, the two input values must also be the same.

One-to-one functions are important because they're the only functions that can have inverses, and functions with inverses aren't all that easy to come by. If a function has an inverse, you can work backward and forward — find an answer if you have a question and find the original question if you know the answer (sort of like *Jeopardy!*).

An example of a one-to-one function is $f(x) = x^3$. The rule for the function involves cubing the variable. The cube of a positive number is positive, and the cube of a negative number is negative. Therefore, every input has a unique output — no other input value gives you that output.

Some functions without the one-to-one designation may look like the previous example, which *is* one-to-one. Take $g(x) = x^3 - x$, for example. This counts as a function because only one output comes with every input. However, the function isn't one-to-one, because you can create the same output or function value from more than one input. For example, $g(1) = (1)^3 - (1) = 1 - 1 = 0$, and $g(-1) = (-1)^3 - (-1) = -1 + 1 = 0$. You have two inputs, 1 and –1, that result in the same output of 0.

Testing for one-to-one functions

You can determine which functions are one-to-one and which are violators by *sleuthing* (guessing and trying), using algebraic techniques and graphing. Most mathematicians prefer the graphing technique because it gives you a nice, visual answer. The basic graphing technique is the *horizontal line test*.

With the horizontal line test, you can see if any horizontal line drawn through the graph cuts through the function more than one time. If a line passes through the graph more than once, the function fails the test and, therefore, isn't a one-to-one function. Figure 5-3 shows a function that passes the horizontal line test and a function that flunks it.

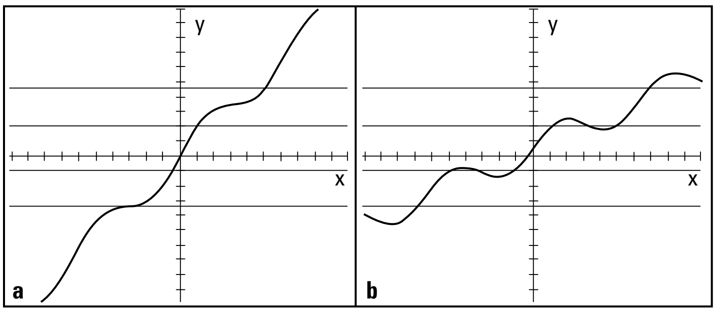

Figure 5-3: The horizontal line test weeds out the one-to-one functions (left) from the violators (right).

Composing Functions

You can perform the basic mathematical operations of addition, subtraction, multiplication, and division on the equations used to describe functions. For example, you can take the two functions $f(x) = x^2 - 3x - 4$ and $g(x) = x + 1$ and perform the four operations on them:

$$f + g = (x^2 - 3x - 4) + (x + 1) = x^2 - 2x - 3$$

$$f - g = (x^2 - 3x - 4) - (x + 1) = x^2 - 3x - 4 - x - 1 = x^2 - 4x - 5$$

$$f \cdot g = \left(x^2 - 3x - 4\right)\left(x + 1\right) = x^3 - 2x^2 - 7x - 4$$

$$\frac{f}{g} = \frac{x^2 - 3x - 4}{x + 1} = \frac{\left(x - 4\right)\cancel{\left(x + 1\right)}}{\cancel{x + 1}} = x - 4$$

Well done, but you have another operation at your disposal — an operation special to functions — called *composition.*

Composing yourself with functions

The *composition* of functions is an operation in which you use one function as the input into another and perform the operations on that input function.

You indicate the composition of functions f and g with a small circle between the function names, $(f \circ g)(x)$, and you define the composition as $(f \circ g)(x) = f(g(x))$.

Here's how you perform an example composition, using the functions $f(x) = x^2 - 3x - 4$ and $g(x) = x + 1$:

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(x+1) \\ &= (x+1)^2 - 3(x+1) - 4 \\ &= x^2 + 2x + 1 - 3x - 3 - 4 \\ &= x^2 - x - 6\end{aligned}$$

The composition of functions isn't commutative. (Addition and multiplication are *commutative,* because you can switch the order and not change the result.) The order in which you perform the composition — which function comes first — matters. The composition $(f \circ g)(x)$ isn't the same as $(g \circ f)(x)$.

Composing with the difference quotient

The *difference quotient* shows up in most high school algebra II classes as an exercise you do after your instructor shows you the composition of functions. You perform this exercise because the difference quotient is the basis of the definition of the derivative in calculus.

So, where does the composition of functions come in? With the difference quotient, you do the composition of some targeted function $f(x)$ and the function $g(x) = x + h$ or $g(x) = x + \Delta x$, depending on what calculus book you use.

The difference quotient for the function *f* is $\dfrac{f(x+h)-f(x)}{h}$.

Perform the difference quotient on $f(x) = x^2 - 3x - 4$.

$$\frac{f(x+h)-f(x)}{h} = \frac{\overbrace{(x+h)^2 - 3(x+h) - 4}^{f(x+h)} - \overbrace{\left(x^2 - 3x - 4\right)}^{f(x)}}{h}$$

Notice that you find the expression for $f(x + h)$ by putting $x + h$ in for every x in the function — $x + h$ is the input variable. Now, continuing on with the simplification:

$$= \frac{x^2 + 2xh + h^2 - 3x - 3h - 4 - x^2 + 3x + 4}{h}$$
$$= \frac{2xh + h^2 - 3h}{h}$$

Did you notice that x^2, $3x$, and 4 all appear in the numerator with their opposites? Now, to finish:

$$= \frac{h(2x + h - 3)}{h}$$
$$= 2x + h - 3$$

Now, this may not look like much to you, but you've created a wonderful result. You've just done some really decent algebra.

Getting Into Inverse Functions

Some functions are *inverses* of one another, but a function can have an inverse only if it's one-to-one. If two functions *are* inverses of one another, each function "undoes" what the other "did." In other words, you use them to get back where you started. The process is sort of like *Jeopardy!* — you have the answer and need to determine the question.

The notation for an inverse function is the exponent –1 written after the function name. The inverse of function $f(x)$, for example, is $f^{-1}(x)$.

Don't confuse the –1 exponent for taking the reciprocal of $f(x)$. The notation is what we're stuck with, so just pay heed.

Here are two inverse functions and how they can "undo" one another:

$$f(x)=\frac{x+3}{x-4} \text{ and } f^{-1}(x)=\frac{4x+3}{x-1}$$

$$f(5)=\frac{5+3}{5-4}=\frac{8}{1}=8 \text{ and } f^{-1}(8)=\frac{4(8)+3}{8-1}=\frac{32+3}{7}=5$$

If you put 5 into function f, you get 8 as a result. If you put 8 into f^{-1}, you get 5 as a result — you're back where you started.

Now, how can you tell when functions are inverses? Read on!

Finding which functions are inverses

In the example from the previous section, I tell you that two functions are inverses and then demonstrate how they work. You can't really *prove* that two functions are inverses by plugging in numbers, however. You may face a situation where a couple numbers work, but, in general, the two functions aren't really inverses.

The only way to be sure that two functions are inverses of one another is to use the following general definition:

> Functions f and f^{-1} are inverses of one another only if $f\left(f^{-1}(x)\right)=x$ *and* $f^{-1}\left(f(x)\right)=x$.

In other words, you have to do the composition in both directions and show that both result in the single value x.

Show that $f(x)=\sqrt[3]{2x-3}+4$ and $g(x)=\frac{(x-4)^3+3}{2}$ are inverses of one another.

First, you perform the composition $f \circ g$:

$$\begin{aligned} f \circ g = f(g) &= \sqrt[3]{2(g)-3}+4 \\ &= \sqrt[3]{\cancel{2}\left(\frac{(x-4)^3+3}{\cancel{2}}\right)-3}+4 \\ &= \sqrt[3]{(x-4)^3+3-3}+4 \\ &= \sqrt[3]{(x-4)^3}+4 \\ &= (x-4)+4 = x \end{aligned}$$

Now you perform the composition in the opposite order:

$$\begin{aligned} g \circ f &= \frac{(f-4)^3+3}{2} \\ &= \frac{\left(\left(\sqrt[3]{2x-3}+4\right)-4\right)^3+3}{2} \\ &= \frac{\left(\sqrt[3]{2x-3}\right)^3+3}{2} \\ &= \frac{(2x-3)+3}{2} \\ &= \frac{2x}{2} \\ &= x \end{aligned}$$

Both come out with a result of x, so the functions are inverses of one another.

Finding an inverse of a function

Up until now in this section, I've given you two functions and told you that they're inverses of one another. I can show you how to create all sorts of inverses for all sorts of one-to-one functions.

Find the inverse of the one-to-one function $f(x)=\frac{x}{x-5}$.

1. **Rewrite the function, replacing *f*(*x*) with *y* to simplify the notation.**

   $$y=\frac{x}{x-5}$$

2. **Change each *y* to an *x* and each *x* to a *y*.**

   $$x=\frac{y}{y-5}$$

3. **Solve for *y*.**

   $$\begin{aligned} x &= \frac{y}{y-5} \\ x(y-5) &= y \\ xy-5x &= y \\ xy-y &= 5x \\ y(x-1) &= 5x,\ y=\frac{5x}{x-1} \end{aligned}$$

4. **Rewrite the function, replacing the *y* with $f^{-1}(x)$.**

   $$f^{-1}(x)=\frac{5x}{x-1}$$